Twist Polynomial Interpolation for Binary Delta-Matroids

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Recap: Delta-Matroids, Rank, and Width

A **set system** is $D = (E, \mathcal{F})$, where E is finite and $\mathcal{F} \subseteq \mathcal{P}(E)$.

Delta-matroid

D is a **delta-matroid** if $\mathcal{F} \neq \varnothing$ and for all $X, Y \in \mathcal{F}$, every $u \in X \triangle Y$, there exists $v \in X \triangle Y$, possibly $v = u$, such that

$$X \triangle \{u, v\} \in \mathcal{F}.$$

$$\mathcal{F}_{\min}(D) = \{F \in \mathcal{F} : |F| \text{ is minimum}\}, \qquad \mathcal{F}_{\max}(D) = \{F \in \mathcal{F} : |F| \text{ is maximum}\}.$$

$$r(D_{\min}) = \min_{F \in \mathcal{F}} |F|, \qquad r(D_{\max}) = \max_{F \in \mathcal{F}} |F|, \qquad w(D) = r(D_{\max}) - r(D_{\min}).$$

Recap: Twists and the Twist Polynomial

For $A \subseteq E$, the **twist** of $D = (E, \mathcal{F})$ by A is

$$D * A = (E, \{A \triangle F : F \in \mathcal{F}\}).$$

$$D * \varnothing = D, \qquad D * E = D^*.$$

The **twist polynomial** is

$$^{\partial}w_D(z) = \sum_{A \subseteq E} z^{w(D*A)}.$$

The coefficient of z^k counts how many twists have width k.

Example: Computing a Twist Polynomial

Let

$$D = \left(\{1,2\},\{\varnothing,\{1\},\{1,2\}\}\right).$$

A	$\mathcal{F}(D*A)$	$w(D*A)$
$\varnothing$	$\{\varnothing,\{1\},\{1,2\}\}$	2
$\{1\}$	$\{\varnothing,\{1\},\{2\}\}$	1
$\{2\}$	$\{\{1\},\{2\},\{1,2\}\}$	1
$\{1,2\}$	$\{\varnothing,\{2\},\{1,2\}\}$	2

$$^{\partial}w_D(z) = 2z + 2z^2.$$

Primal Types

Let

$$\mathcal{F}_{\min+i}(D) = \{F \in \mathcal{F} : |F| = r(D_{\min}) + i\}.$$

For $e \in E$, the **primal type** of e is:

Type	Condition
p	Some minimum feasible set contains e.
u	No set in $\mathcal{F}_{\min}(D) \cup \mathcal{F}_{\min+1}(D)$ contains e.
t	No minimum feasible set contains e, but some set in $\mathcal{F}_{\min+1}(D)$ contains e.

Dual Type

The **dual type** of e in D is the primal type of e in

$$D^* = D * E.$$

So each element has a two-letter type:

$$pp,\ pu,\ pt,\ up,\ uu,\ ut,\ tp,\ tu,\ tt.$$

For example, type ut means primal type u in D, and primal type t in D^*.

Width Change Under a Single Twist

The type of e determines the change in width under $D \mapsto D * e$.

Type of e	$w(D * e) - w(D)$
pp	$+2$
uu	-2
$pt,\ tp$	$+1$
$ut,\ tu$	-1
$pu,\ up,\ tt$	0

Therefore,

$$|w(D * e) - w(D)| \leq 2.$$

Binary Delta-Matroids

Let C be a symmetric $|E| \times |E|$ matrix over $\mathbb{Z}/2\mathbb{Z}$, with rows and columns indexed by E.
Define

$$D(C) = (E, \mathcal{F}), \qquad \mathcal{F} = \{A \subseteq E : C[A] \text{ is nonsingular}\}.$$

Here $C[A]$ is the principal submatrix indexed by A, and $C[\varnothing]$ is nonsingular by convention.
A delta-matroid D is **binary** if some twist of D is isomorphic to $D(C)$.
Equivalently, there exists $A \subseteq E$ such that

$$D * A \cong D(C).$$

This binary condition is essential in the interpolation theorem.

Example of a Binary Delta-Matroid

Let

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

over $\mathbb{Z}/2\mathbb{Z}$, with $E = \{1, 2\}$.

A	$C[A]$	$\det(C[A])$
$\varnothing$	$C[\varnothing]$	1
$\{1\}$	(1)	1
$\{2\}$	(0)	0
$\{1, 2\}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	1

Thus

$$D(C) = \big(\{1, 2\}, \{\varnothing, \{1\}, \{1, 2\}\}\big).$$

How Many Nonequivalent Delta-Matroids?

Here "nonequivalent" means up to relabeling of E and twisting.

$\lvert E\rvert$	Delta-matroids	Binary	Nonbinary
2	5	5	0
3	16	13	3
4	90	40	50
5	2902	141	2761

Note

The total number of nonequivalent delta-matroids grows doubly exponentially in $|E| = n$.

Even/Odd Interpolating Polynomials

Let

$$p(z) = \sum_i c_i z^i.$$

Write

$$p(z) = p_e(z^2) + z p_o(z^2),$$

where p_e records the even-degree terms and zp_o records the odd-degree terms.

- $p(z)$ is **even** if only even powers appear.
- $p(z)$ is **odd** if only odd powers appear.
- $p(z)$ is **even-interpolating** if its even-degree support has no gaps.
- $p(z)$ is **odd-interpolating** if its odd-degree support has no gaps.

$$1 + 3z^2 + 5z^4 \quad \text{is even-interpolating.}$$

$$1 + 3z^2 + 5z^6 \quad \text{has an even gap.}$$

Main Theorem

Theorem (Zhao–Yan)

Let $D = (E, \mathcal{F})$ be a **binary delta-matroid**. *Then its twist polynomial*

$$^{\partial}w_D(z) = \sum_{A \subseteq E} z^{w(D*A)}$$

is either:

1. *an even polynomial,*
2. *an odd polynomial, or*
3. *both even-interpolating and odd-interpolating.*

Loops, Coloops, and Deletion

Let $D = (E, \mathcal{F})$ be a set system and $e \in E$.

- e is a **loop** if $e \notin F$ for every $F \in \mathcal{F}$.
- e is a **coloop** if $e \in F$ for every $F \in \mathcal{F}$.

The deletion $D - e$ is defined by

$$D - e = (E - e, \mathcal{F}')$$

where

$$\mathcal{F}' = \begin{cases} \{F \in \mathcal{F} : F \subseteq E - e\}, & e \text{ is not a coloop,} \\ \{F - e : F \in \mathcal{F}\}, & e \text{ is a coloop.} \end{cases}$$

Lemma 14: Primal Type Is Stable Under Deletion

Lemma

Let $D = (E, \mathcal{F})$ be a delta-matroid and let $e_1, e_2 \in E$ with $e_1 \neq e_2$. Then the primal type of e_2 in $D - e_1$ is the same as its primal type in D.

One sample case: suppose e_1 is a coloop and e_2 has primal type p.
Since e_1 is a coloop,

$$\mathcal{F}_{\min}(D - e_1) = \{F - e_1 : F \in \mathcal{F}_{\min}(D)\}.$$

Because e_2 has primal type p, there exists

$$F \in \mathcal{F}_{\min}(D) \quad \text{with} \quad e_2 \in F.$$

Then

$$F - e_1 \in \mathcal{F}_{\min}(D - e_1) \quad \text{and} \quad e_2 \in F - e_1.$$

So e_2 still has primal type p in $D - e_1$.

Even and Odd Delta-Matroids

A delta-matroid $D = (E, \mathcal{F})$ is **even** if all feasible sets have the same parity of cardinality. Equivalently,

$$|X \triangle Y| \text{ is even for all } X, Y \in \mathcal{F}.$$

Otherwise, D is **odd**.

$$\mathcal{F} = \{\varnothing, \{1, 2\}\} \quad \Rightarrow \quad D \text{ is even.}$$

$$\mathcal{F} = \{\varnothing, \{1\}, \{1, 2\}\} \quad \Rightarrow \quad D \text{ is odd.}$$

Lemma 16: The Binary Requirement

Lemma

Let $D = (E, \mathcal{F})$ be a **binary** *delta-matroid. If every element of E has primal type u in D, then D is even.*

Proof idea: If every element has primal type u, then no singleton is feasible. For a normal binary delta-matroid $D = D(C)$, this means the representing symmetric matrix C over $\mathbb{Z}/2\mathbb{Z}$ has zero diagonal.
But an odd-sized symmetric matrix over $\mathbb{Z}/2\mathbb{Z}$ with zero diagonal cannot be nonsingular. So no odd feasible set exists, and D is even.

Lemma 18: Twists Commute with Deletion

Lemma

Let $D = (E, \mathcal{F})$ be a delta-matroid and let $A, B \subseteq E$ with

$$A \cap B = \varnothing.$$

Then

$$(D * A) - B = (D - B) * A.$$

Theorem 19

Theorem

Let $D = (E, \mathcal{F})$ be a binary delta-matroid.
If the twist width set

$$W(D) = \{w(D * A) : A \subseteq E\}$$

contains two consecutive integers k and $k+1$, then every integer

$$m \quad \text{with} \quad k+2 \leq m \leq w_M(D)$$

is also a twist width.

Here

$$w_M(D) = \max\{w(D * A) : A \subseteq E\}.$$

So once two adjacent widths occur, every larger width up to the maximum must occur.

Theorem 19 Proof

Theorem

Let D be binary. If k and $k+1$ are twist widths of D, then every

$$k+2 \le m \le w_M(D)$$

is also a twist width.

Choose $A \subseteq E$ such that

$$w(D * A) = k.$$

It is enough to show that $k+2$ occurs. Then we repeat the argument for

$$(k+1, k+2), \quad (k+2, k+3), \quad \ldots$$

Assume, for contradiction, that $k+2$ does not occur. If $w_M(D) \le k+1$, there is nothing to prove. Otherwise,

$$w_M(D) \ge k+3.$$

Twist one element at a time toward a width at least $k+3$. Since each step changes width by at most 2, the first width at least $k+3$ must be exactly $k+3$.

Theorem 19 Proof Continued

Choose $B \subseteq E$ of minimum cardinality such that

$$w(D * A * B) = k + 3.$$

Set

$$D' = D * A * B.$$

For $e \in B$,

$$D' * e = D * A * (B - \{e\}).$$

So twisting e in D' is the same as removing e from the chosen twisting set B. Now use the width-change table. If e has type

$$tt, \; pu, \; \text{or } up,$$

then

$$w(D' * e) = k + 3.$$

Thus $B - \{e\}$ already gives width $k + 3$, contradicting the minimality of B.

Theorem 19 Proof Continued

The other types are eliminated as follows.
If e has type pp, pt, or tp, then

$$w(D' * e) \geq k + 4.$$

But $D' * e = D * A * (B - \{e\})$. Starting from width k, twist the elements of $B - \{e\}$ one at a time.
Since each step changes width by at most 2, and $k + 2$ is assumed absent, this path must hit $k + 3$ before reaching $k + 4$ or higher.
That gives a proper subset of B producing width $k + 3$, again contradicting minimality.
If e has type ut or tu, then

$$w(D' * e) = k + 2,$$

contradicting the assumption that $k + 2$ is absent.
Therefore every element of B has type

$$uu$$

in D'.

Theorem 19 Proof Continued

Delete all elements outside B:

$$N = D' - (E - B).$$

By Lemma 14, deletion preserves the primal type of elements in B. Hence every element of N has primal type u.
Since N is binary, Lemma 16 implies that N is even.
Now twist the elements of B back one at a time. Lemmas 14 and 18 let us check the relevant types inside twists of N. Since N is even, no element has primal or dual type t. So every step has type uu, up, pu, or pp, and the width change is one of

$$-2,\ 0,\ 0,\ +2.$$

Thus the total change must be even. But twisting all of B sends

$$D' = D * A * B \quad \text{back to} \quad D * A,$$

so the total change is

$$k - (k+3) = -3,$$

an odd number. Contradiction.
Therefore $k+2$ must occur.

Proof of the Main Theorem

Let
$$W(D) = \{w(D * A) : A \subseteq E\} = \{w_1 < w_2 < \cdots < w_t\}.$$
Since a single twist changes width by at most 2,
$$w_{i+1} - w_i \leq 2.$$

Case 1: all widths have the same parity.
Then the gaps are at most 2, so the widths form consecutive values of one parity.
Therefore the twist polynomial is either even or odd.
Case 2: both parities occur.
At the first place where the parity changes, two consecutive widths occur:
$$k, \quad k+1.$$

By Theorem 19, every larger width up to $w_M(D)$ occurs.
Thus both the even-degree and odd-degree parts have no gaps.

$${}^{\partial}w_D(z) \text{ is both even-interpolating and odd-interpolating.}$$

Ribbon Graphs and Delta-Matroids

For a ribbon graph G, define

$$D(G) = \big(E(G), \mathcal{F}(G)\big),$$

where

$$\mathcal{F}(G) = \{F \subseteq E(G) : (V(G), F) \text{ is a spanning quasi-tree}\}.$$

A **spanning quasi-tree** contains every vertex of G and has exactly one boundary component in each connected component.

Ribbon-graphic delta-matroid

The set system $D(G)$ is a delta-matroid. Moreover,

$$D(G) \text{ is binary.}$$

Why $D(G)$ Is Binary: Proof Sketch

Choose a spanning quasi-tree Q in each connected component of G, and form the partial dual

$$H = G^Q.$$

Each component of H has one vertex. For a one-vertex ribbon graph, define a symmetric matrix C over $\mathbb{Z}/2\mathbb{Z}$ by

$$C_{ee} = \begin{cases} 1, & e \text{ is non-orientable,} \\ 0, & e \text{ is orientable,} \end{cases}$$

and, for $e \neq f$,

$$C_{ef} = 1 \quad \Longleftrightarrow \quad e \text{ and } f \text{ are interlaced around the vertex.}$$

Bouchet's representation gives $D(H) = D(C)$.
Since partial duality corresponds to twisting,

$$D(H) = D(G^Q) = D(G) * Q.$$

Thus a twist of $D(G)$ is represented by a symmetric matrix over $\mathbb{Z}/2\mathbb{Z}$, so

$$D(G) \text{ is binary.}$$

Euler Genus and Width

For a ribbon graph G, let

$$v(G),\ e(G),\ f(G),\ c(G)$$

denote the number of vertices, edges, boundary components, and connected components.
The Euler characteristic is

$$\chi(G) = v(G) - e(G) + f(G).$$

The **Euler genus** is

$$\varepsilon(G) = 2c(G) - \chi(G).$$

A past result is

$$\boxed{\varepsilon(G) = w(D(G)).}$$

Thus Euler genus for ribbon graphs corresponds to width for delta-matroids.

Theorem 19 for Ribbon Graphs

We also have

$$D(G^A) = D(G) * A, \qquad \varepsilon(G^A) = w(D(G) * A).$$

Since $D(G)$ is binary, Theorem 19 applies.

Theorem

If

$$\{\varepsilon(G^A) : A \subseteq E(G)\}$$

contains two consecutive integers k *and* $k+1$*, then it contains every integer*

$$k + 2 \leq m \leq \varepsilon_M(G),$$

where

$$\varepsilon_M(G) = \max_{A \subseteq E(G)} \varepsilon(G^A).$$

So the partial-dual polynomial has the same even/odd interpolation behavior.

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